ANALYSIS OF THE TWO-DIMENSIONAL TEMPERATURE FIELD OF A BOUNDED CYLINDER WITH A PLANAR INTERNAL HEAT SOURCE OF CONSTANT STRENGTH WITH BOUNDARY CONDITIONS OF THE FIRST KIND

V. P. Kozlov, A. G. Shashkov,
and G. M. Volokhov

UDC 536.21

We obtain the two-dimensional nonstationary solution for a bounded cylinder with an internal source of constant strength for time-varying boundary conditions of the first kind. Optimal dimensions of a specimen are determined on a theoretical basis (for the case involving application to plates), and these agree, with sufficient accuracy, with those obtained from the one-dimensional solutions. Thermophysical characteristics can be determined from the two-dimensional computational relationships which are supplied.

One of the most involved problems in theoretical and applied studies of the thermophysical characteristics of materials is that of estimating the reliability of the results obtained. A fundamental source of systematic errors, arising in applying these or other methods for determining thermophysical characteristics, consists in the fact that the majority of them are based on solutions of the one-dimensional differential equation of heat conduction with specific initial and boundary conditions. In actuality, the theoretical premises of unboundedness of the experimental specimens are not fulfilled.

The selection of optimal relationships among the dimensions of a test specimen, such that the one-dimensionality of the temperature field is preserved with a specified accuracy without applying any adjustment factors, is a problem associated with the solution of two- and three-dimensional heat conduction problems. Papers [1-5] have been devoted to this very problem. In [3] an analysis was given of the two-dimensional temperature field of a hollow cylinder with boundary conditions of the first, second, and third kinds (using a quasistationary method). In [4, 5] the two-dimensional temperature field of a solid cylinder was obtained for combined boundary conditions constant with time.

The aim of the present paper is to furnish a theoretical basis for optimizing the dimensions of a specimen (for the case involving application to plates) when stationary and quasistationary methods are used for determining the thermophysical characteristics of materials.

We are given a bounded cylinder of height 2h and diameter 2R (coordinate origin at the center), which is initially in thermal equilibrium with the surrounding medium, i.e., the temperature of the cylinder is equal to the temperature T_0 of the surrounding medium. At the initial instant a source of specific power q begins to operate in the central plane (z = 0), the lateral surface begins to heat up at the constant rate b_2, and the end surfaces begin to heat up at the constant rate b_1. It is required to find an expression for the temperature field of the bounded cylinder, i.e., we seek to solve the differential equation of heat conduction

$$\frac{1}{a}\frac{\partial T}{\partial \tau} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} \tag{1}$$

for the following initial and boundary conditions:

$$T(r,\ z,\ 0) = T_0 = \text{const},$$

Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 18, No. 4, pp. 720-726, April, 1970. Original article submitted September 8, 1969.

$$
\begin{aligned}
&T(r,\ h,\ \tau) = T_0 + b_1\tau,\\
&T(R,\ z,\ \tau) = T_0 + b_2\tau,\\
&-\lambda \frac{\partial T(r,\ 0,\ \tau)}{\partial z} = q,\\
&\frac{\partial T(0,\ z,\ \tau)}{\partial r} = 0.
\end{aligned}
\tag{2}
$$

The solution of the given problem, obtained by the method of Hankel and Laplace transformations, can be represented in the form

$$
\begin{aligned}
\frac{\theta}{\mathrm{Pd}_h} = {} & \mathrm{Fo}_h \left[c_B + 2(1-c_B) \sum_{n=1}^{\infty} \frac{I_0\left(\mu_n \frac{r}{R}\right) \mathrm{ch}\,\mu_n \frac{z}{R}}{\mu_n I_1(\mu_n)\, \mathrm{ch}\,\mu_n k} \right] \\
& - \frac{c_B}{4k^2} \left(1 - \frac{r^2}{R^2} - 8 \sum_{n=1}^{\infty} \frac{I_0\left(\mu_n \frac{r}{R}\right) \mathrm{ch}\,\mu_n \frac{z}{R}}{\mu_n^3 I_1(\mu_n)\, \mathrm{ch}\,\mu_n k} \right) \\
& + \frac{z}{h}\, \frac{1-c_B}{k} \sum_{n=1}^{\infty} \frac{I_0\left(\mu_n \frac{r}{R}\right) \mathrm{sh}\,\mu_n \frac{z}{R}}{\mu_n^2 I_1(\mu_n)\, \mathrm{ch}\,\mu_n k} - \frac{1-c_B}{k} \sum_{n=1}^{\infty} \frac{I_0\left(\mu_n \frac{r}{R}\right) \mathrm{ch}\,\mu_n \frac{z}{R}\, \mathrm{sh}\,\mu_n k}{\mu_n^2 I_1(\mu_n)\, \mathrm{ch}^2 \mu_n k} \\
& + 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} (\lambda_m^2 + c_B \mu_n^2 k^2)\, I_0\left(\mu_n \frac{r}{R}\right) \cos \lambda_m \frac{z}{h}}{\mu_n I_1(\mu_n)\, \lambda_m (\lambda_m^2 + \mu_n^2 k^2)^2} \\
& \times \exp\left[-(\lambda_m^2 + \mu_n^2 k^2)\right] \mathrm{Fo}_h + \frac{2\,\mathrm{Ki}_h}{k\,\mathrm{Pd}_h} \sum_{n=1}^{\infty} \frac{I_0\left(\mu_n \frac{r}{R}\right) \mathrm{sh}\,\mu_n \left(k - \frac{z}{R}\right)}{\mu_n^2 I_1(\mu_n)\, \mathrm{ch}\,\mu_n k} \\
& - 4 \frac{\mathrm{Ki}_h}{\mathrm{Pd}_h} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} I_0\left(\mu_n \frac{r}{R}\right) \sin \lambda_m \left(1 - \frac{z}{h}\right)}{\mu_n I_1(\mu_n) (\lambda_m^2 + \mu_n^2 k^2)} \exp\left[-(\lambda_m^2 + \mu_n^2 k^2)\right] \mathrm{Fo}_h.
\end{aligned}
\tag{3}
$$

Equation (3) has its simplest form when $b_1 = b_2 = 0$. In this case the surfaces of the bounded cylinder are maintained at a constant temperature, equal to the initial temperature, i.e.,

$$
\begin{aligned}
\theta = {} & \mathrm{Ki}_h \frac{2}{k} \sum_{n=1}^{\infty} \frac{I_0\left(\mu_n \frac{r}{R}\right) \mathrm{sh}\,\mu_n \left(k - \frac{z}{R}\right)}{\mu_n^2 I_1(\mu_n)\, \mathrm{ch}\,\mu_n k} \\
& - 4\,\mathrm{Ki}_h \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} I_0\left(\mu_n \frac{r}{R}\right) \sin \lambda_m \left(1 - \frac{z}{h}\right)}{\mu_n I_1(\mu_n) (\lambda_m^2 + \mu_n^2 k^2)} \exp\left[-(\lambda_m^2 + \mu_n^2 k^2)\right] \mathrm{Fo}_h.
\end{aligned}
\tag{4}
$$

Figure 1 displays curves showing the variation of θ/Ki_h with the Fourier number Fo_h, calculations being made for the center of the bounded cylinder for various ratios of the parameter k. All calculations were made on the electronic digital computer "Promin'." These curves may be compared with curve 1, constructed for the center of an unbounded plate [6]. The general solution for an unbounded plate is readily obtained from expression (4) by letting $R \to \infty$. In the stationary thermal state, approached theoretically as $\mathrm{Fo}_h \to \infty$ (with an accuracy of 0.6%, in practice, when $\mathrm{Fo}_h = 2$, with relation to the ideal stationary state), the deviation of the corresponding function θ/Ki_h from the curve 1 reaches its maximum value, wherein the magnitude of this deviation depends on the value of the parameter k (see Fig. 1). We have taken the most unfavorable case of heat transfer on the lateral surface of the plate ($\mathrm{Bi} \to \infty$). For smaller values of the Biot number these deviations are less [4]. From Fig. 1 it is evident that the use of specimens with a parameter value of $k = 1/4$ will guarantee sufficient accuracy toward satisfying the condition of unboundedness of the specimen in its central region.

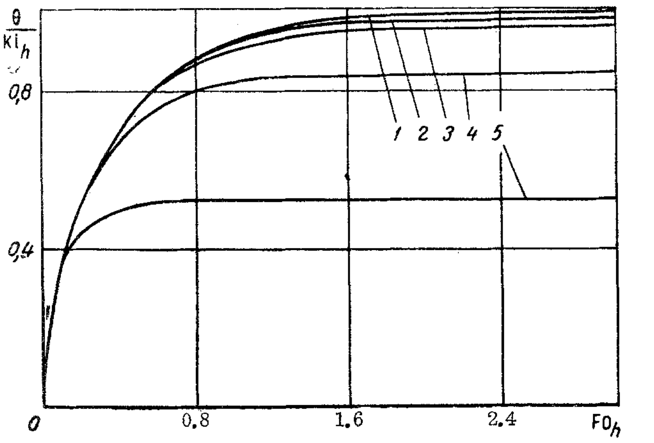

Fig. 1. Variation of θ/Ki_h with the Fourier number Fo_h at the center of the bounded cylinder for various values of k: 1) center of an unbounded plate and k = 1/5 (with an accuracy of 0.8%); 2) k = 1/4 (deviation from curve 1 is 1.4%); 3) k = 1/3 (deviation from curve 1 is 4.2%); 4) k = 1/2 (deviation from curve 1 is 15%); 5) k = 1 (deviation from curve 1 is 52%).

Before going on to consider the more general case $b_1 \neq b_2 \neq 0$, $q \neq 0$, it is appropriate to analyze the temperature field of the unbounded plate. The general solution for the unbounded plate may be obtained from the solution (3) by putting $b_2 = 0$ ($c_B = 0$) and letting $R \to \infty$. This solution coincides completely with the solution given in [6], and for $\mathrm{Bi} \to \infty$ it has the form

$$\frac{\theta}{\mathrm{Pd}_h} = \mathrm{Fo}_h - \frac{1}{2}\left(1 - \frac{z^2}{h^2}\right) + \frac{\mathrm{Ki}_h}{\mathrm{Pd}_h}\left(1 - \frac{z}{h}\right) + 2\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{\lambda_m^3} \cos \lambda_m \frac{z}{h} \exp(-\lambda_m^2 \mathrm{Fo}_h) - 2\frac{\mathrm{Ki}_h}{\mathrm{Pd}_h} \sum_{m=1}^{\infty} \frac{1}{\lambda_m^2} \cos \lambda_m \frac{z}{h} \exp(-\lambda_m^2 \mathrm{Fo}_h). \quad (5)$$

Figure 2 shows how the generalized function $\theta/\mathrm{Pd}_h\mathrm{Fo}_h$ varies with the Fourier number Fo_h at the center of an unbounded plate (the plane in which a source of constant strength is operative) for various ratios of the Kirpichev and Predvoditelev numbers formulated on the basis of Eq. (5). The parameter $\mathrm{Ki}_h/\mathrm{Pd}_h$ characterizes the relationship between the specific thermal flow at the center and that on the surface of the plate in the quasistationary thermal regime. It is not difficult to show that for $\mathrm{Ki}_h/\mathrm{Pd}_h = 0.5$ the indicated flows are equal (the regime of initial adiabatic heating of the plate: curve 4). The temperature distribution through the plate thickness in this case is parabolic, the vertex of the parabola being located at $z = h/2$. For the values $\mathrm{Ki}_h/\mathrm{Pd}_h < 0.5$, we have the nonsymmetric heating condition in which the vertex of the parabola is displaced, relative to the point $z = h/2$, toward the center as the ratio $\mathrm{Ki}_h/\mathrm{Pd}_h$ is made smaller. In this case, to determine the thermophysical characteristics it is necessary to record temperatures at three points of the specimen. The case for which the ratio $\mathrm{Ki}_h/\mathrm{Pd}_h > 0.5$ is of no practical interest since then the thermal flow in the plane $z = 0$ will predominate over the thermal flow formed on the surface of the plate, and the quasistationary regime appears for sufficiently large values of the Fourier number. From Fig. 2 it is evident that for various values of the ratio $\mathrm{Ki}_h/\mathrm{Pd}_h$, the generalized function $\theta/\mathrm{Pd}_h\mathrm{Fo}_h \to 1$ as $\mathrm{Fo}_h \to \infty$. In accordance with this, the time of appearance of the quasistationary thermal state varies. Thus, for example, the quasistationary regime, with an accuracy within 1%, for values of the ratio $\mathrm{Ki}_h/\mathrm{Pd}_h$ equal to 0.5, 0.2, 0.1, 0.001, sets in for Fourier number Fo_h values equal, respectively, to 1, 1.4, 1.5, 1.6. Consequently, in the case of initially adiabatic heating the quasistationary regime sets in much sooner in comparison with the nonsymmetric heating of the plate. The way in which the generalized function $\theta/\mathrm{Pd}_h\mathrm{Fo}_h$ varies at the center of the plate for small Fourier numbers can be explained by a change in the rate of heating in the regime preceding the quasistationary regime.

In Fig. 3 we present two-dimensional curves showing the variation of the function $\theta/\mathrm{Pd}_h\mathrm{Fo}_h$ with the Fourier number Fo_h at the center of a bounded cylinder (disk) for various values of c_B and k, the curves being based on the solution (3). The plots are drawn for the ratio $\mathrm{Ki}_h/\mathrm{Pd}_h = 0.1$. A comparison may be made of these curves with the one-dimensional curve 4. The maximum deviation between the one- and two-dimensional generalized functions occurs for the stationary state (quasistationary regime). For specimens with the ratio k = 1/4, 1/3, 1/2, 1, the magnitude of these deviations in the most unfavorable condition of the experiment, $c_B = 0$ (the case corresponding to heat transfer on the lateral surface of the bounded cylinder when $\mathrm{Bi} \to \infty$) amounts to 0.85, 3.67, 19, 68%, respectively, for $\mathrm{Fo}_h = 3$. With increasing Fourier number the size of these deviations increases. However for k = 1/4 this deviation amounts to 1.6% for $\mathrm{Fo}_h = 10$. For $c_B = 1$ these deviations are equal to 0.2, 1.4, 3.1, 9.7%, respectively, and remain constant as the Fourier number increases. For other values of c_B these deviations, for a specific k-ratio, will lie between the limits mentioned above.

Thus by studying specimens in the form of plates with a ratio of k = 1/4, we can guarantee sufficient accuracy in satisfying the infinite plate condition at its central region when use is made of the quasistationary method for determining the thermophysical characteristics of materials.

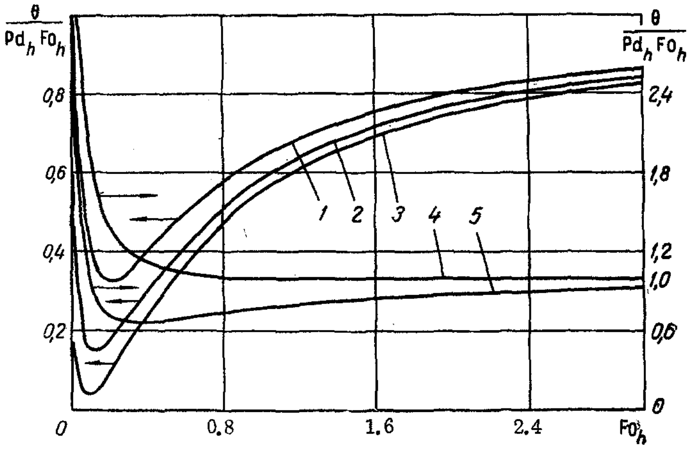

Fig. 2. Variation of θ/Pd_hFo_h with the Fourier number Fo_h for various values of Ki_h/Pd_h at the center of an unbounded plate: 1) $Ki_h/Pd_h = 0.1$; 2) 0.04; 3) 0.01; 4) 0.5; 5) 0.25.

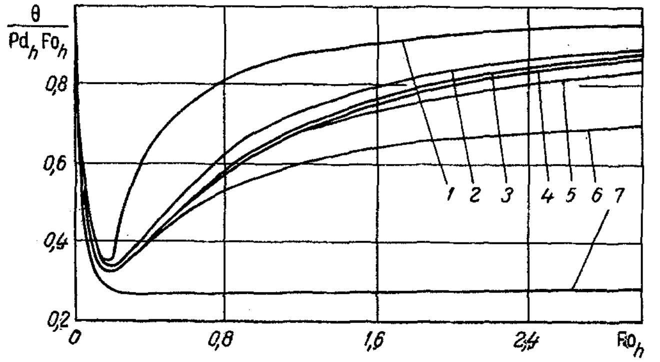

Fig. 3. Variation of θ/Pd_hFo_h with the Fourier number Fo_h for various values of c_B and k at the center of a bounded cylinder (disk): 1) $k = 1$, $c_B = 1$; 2) $k = 1/2$, $c_B = 1$; 3) $k = 1/3$, $c_B = 1$; 4) center of unbounded plate and $k = 1/4$, $c_B = 0$ (within 0.85%); 5) $k = 1/3$, $c_B = 0$; 6) $k = 1/2$, $c_B = 0$; 7) $k = 1$, $c_B = 0$ ($Ki/Pd_h = 0.1$).

TABLE 1. Values of the Constants $C_1(k)$, $C_2(k)$, $C_3(k)$, $C_4(k)$

k	$C_1(k)$	$C_2(k)$	$C_3(k)$	$C_4(k)$
1	1,7412	2,5748	0,7579	3,9430
1/2	1,1159	2,0827	0,9516	3,9674
1/3	1,0268	2,0215	0,9902	3,9973
1/4	1,0054	2,0070	0,9933	3,9995

It is a known fact that in the quasistationary thermal regime with an internal heat source of constant strength, both the thermal diffusivity and thermal conductivity of a material can be determined. We now obtain two-dimensional computational formulas for the indicated characteristics for the case in which the heating rates on the ends and on the lateral surface are equal. We assume, for simplicity, that the temperature is measured at the points with coordinates $r = 0$, $z = 0$; $r = 0$, $z = h/2$; $r = 0$, $z = h$. Upon finding the value of the corresponding difference of temperatures from expression (3) for $b_1 = b_2 = b$ between the two points indicated, and then solving these equations simultaneously, we obtain

$$a = \frac{bh^2}{4\left[C_1(k)\,\Delta T_{h,0} - C_2(k)\,\Delta T_{\frac{h}{2},0}\right]}, \tag{6}$$

$$\lambda = \frac{qh}{C_3(k)\,\Delta T_{h,0} - C_4(k)\,\Delta T_{\frac{h}{2},0}}, \tag{7}$$

where $\Delta T_{h,0}$, $\Delta T_{h/2,0}$ are, respectively, the differences of temperatures between the points with coordinates $r = 0$, $z = h$; $r = 0$, $z = 0$ and $r = 0$, $z = h/2$; $r = 0$, $z = 0$. $C_1(k)$, $C_2(k)$, $C_3(k)$, $C_4(k)$ are constants for a given k-ratio. Values of these constants, obtained on the electronic digital computer "Promin'" for certain k-ratios, are shown in Table 1. The corresponding one-dimensional formulas may be found in [5].

From Table 1 it is evident that for $k = 1/4$ the values of the constants appearing before the corresponding temperature drops are, with sufficient accuracy, close to the one-dimensional values. In conclusion, it should be noted that in the quasistationary thermal regime assignment of a constant heating rate on the plate surfaces is equivalent to the assignment of a constant thermal flow. Therefore all the conclusions relating to the character of a change in the temperature fields hold for this case also.

NOTATION

T, T_0	are the temperature of any point of finite cylinder and initial temperature;
a	is the thermal diffusivity;
λ	is the thermal conductivity;
R	is the cylinder radius;
h	is the half-height of cylinder;
b_1, b_2	are the rate of heating on end face and side of finite cylinder, deg/sec;
r, z	are the current coordinates of finite cylinder;
τ	is the time;
I_0, I_1	are the Bessel functions of zero and first order of first kind;
μ_n	are the roots of Bessel function of zeroth order of first kind;
$k = h/R$	is the parameter characterizing relationship between height and diameter of cylinder;
$Fo_h = a\tau/h^2$	is the Fourier number;
$Pd_h = b_1h^2/aT_0$	is the Predvoditelev number;
$Ki_h = qh/\lambda T_0$	is the Kirpichev number;
c_B	is the parameter characterizing relationship between heating rates on cylinder surfaces;
q	is the specific heat flux in plane ($z = 0$);

$\lambda_m = (2m - 1)(\pi/2)$;
$\theta = (T - T_0)/T_0$.

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